The Inverse 1-Median Problem in $\mathbb{R}^d$ with the Chebyshev-Norm ⋆

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1 Introduction

This paper focuses on the weighted 1-median problem in $\mathbb{R}^d$ where the distance of two points is measured by the Chebyshev-norm. So far this problem is only well understood for $d = 2$. In this case, a linear time algorithm is given in Hamacher [2]. In this note, we give the first combinatorial algorithm for $d \geq 3$. Furthermore, we discuss an optimality criterion for the d-dimensional case which is based on linear programming. Using this optimality criterion we are able to solve the inverse location problem. In the inverse problem the facility is already given and the task is to modify the weights of the points at minimum cost such that the given facility is a 1-median with respect to the new weights.

2 The 1-median problem

The 1-median problem discussed in this note is defined as follows: Given n points $P_1, \ldots, P_n$ with $P_i = (x_1^i, \ldots, x_d^i) \in \mathbb{R}^d$ for $i = 1, \ldots, n$ and associated non-negative weights w_i the task is to find a point $P^* = (x_1^*, \ldots, x_d^*) \in \mathbb{R}^d$

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such that

$$\sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty \geq \sum_{i=1}^{n} w_i \, \|P_i - P^*\|_\infty$$

for all $P \in \mathbb{R}^d$. Note that using some straightforward techniques the problem $\min_{P=(y_1,\dots,y_d)\in\mathbb{R}^d} \sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty$ can be written as a linear programming problem in the following form:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} w_i \, z_i && && (1)\\
\text{s.t.} \quad & z_i + y_j \geq x_j^i && i = 1,\dots,n, \;\; j = 1,\dots,d && (2)\\
& z_i - y_j \geq -x_j^i && i = 1,\dots,n, \;\; j = 1,\dots,d && (3)\\
& y_j \in \mathbb{R}, \;\; z_i \in \mathbb{R} && j = 1,\dots,d, \;\; i = 1,\dots,n. && (4)
\end{aligned}$$

Due to the fact that the variables y_j do not appear in the objective function, we use the well known Fourier-Motzkin elimination to get rid of these variables in the constraints. We obtain the following equivalent problem

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} w_i \, z_i \\
\text{s.t.} \quad & z_i + z_k \geq d^{ik} && i = 1,\dots,n \;\; k = 1,\dots,n\\
& z_i \geq 0 && i = 1,\dots,n
\end{aligned}$$

where $d^{ik} := \max_j |x_j^i - x_j^k|$. It is easy to see that the corresponding dual problem is the linear relaxation of the maximum-weight-b-matching problem on a complete graph. It is shown in Antsee [1] that this graphtheoretical problem can be solved by a min-cost-flow problem in a bipartite graph.

3 The Inverse Problem

An instance of the inverse problem is given by a set of n points $P_1, \dots, P_n \in \mathbb{R}^d$ with corresponding non-negative weights $w_i \geq 0$ and a point P_0 (which may coincide with a given point). The task is to find new weights $\tilde{w}_i \geq 0$ such that P_0 is a 1-median with respect to $\tilde{w}_i$ and $\|w - \tilde{w}\|_1$ is minimized. The problem can be formulated in a compact form as follows:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} |w_i - \tilde{w}_i| \\
\text{s.t.} \quad & \sum_{i=1}^{n} \tilde{w}_i \, \|P_i - P\|_\infty \geq \sum_{i=1}^{n} \tilde{w}_i \, \|P_i - P_0\|_\infty && \forall P \in \mathbb{R}^d\\
& \tilde{w}_i \geq 0 && i = 1,\dots,n.
\end{aligned}$$

Before we give a combinatorial algorithm we state the following lemma.

Lemma 1 *There exists an optimal solution w^* of the inverse location problem such that $w_i \geq w_i^*$ holds for all $i = 1, \ldots, n$.*

Let us consider the dual linear programming problem of (1)—(4). We introduce non-negative dual variables $u_{i,j}$ for the constraints (2) and non-negative dual variables $v_{i,j}$ for the constraints (3) and obtain

$$\begin{array}{lll} \max & \sum_{i=1}^{n} \sum_{j=1}^{d} x_j^i \,(u_{i,j} - v_{i,j}) & \\ \text{s.t.} & \sum_{j=1}^{d} (u_{i,j} + v_{i,j}) = w_i & i = 1, \ldots, n \\ & \sum_{i=1}^{n} u_{i,j} = \sum_{i=1}^{n} v_{i,j} & j = 1, \ldots, d \\ & u_{i,j} \geq 0, \;\; v_{i,j} \geq 0 & i = 1, \ldots, n, \;\; j = 1, \ldots, d. \end{array}$$

In order to interpret the dual problem let us construct the following flow problem: Consider the bipartite graph $G = (V_1 \cup V_2, E)$ where the set V_1 consists of n vertices, one for each point P_i of the 1-median problem. The set V_2 has exactly $2d$ vertices representing the closed cones

$$Q_j^{\geq} := \{x \in \mathbb{R}^d : x_j \geq 0 \text{ and } |x_j| \geq |x_k| \, \forall k = 1, \ldots, d\}$$

and

$$Q_j^{\leq} := \{x \in \mathbb{R}^d : x_j \leq 0 \text{ and } |x_j| \geq |x_k| \, \forall k = 1, \ldots, d\}$$

for $j = 1, \ldots, d$. We have an edge $(P_i, Q_j^{\sim})$ if P_i is in the cone $Q_j^{\sim}$ ($\sim \in \{\leq, \geq\}$). Moreover, we set the capacity $u(e)$ of the edges of the bipartite graph to infinity. Furthermore, we add a source s and the edges (s, P_i) for all $i = 1, \ldots, n$ with $u(s, P_i) = w_i$. Finally, we introduce a sink t and an edge from each vertex in V_2 to t with infinite capacity. We denote this network by $I(P_1, \ldots, P_n, w)$. Furthermore, a flow in $I(P_1, \ldots, P_n, w)$ is called perfect if the flow on the edges $(Q_j^{\geq}, t)$ and $(Q_j^{\leq}, t)$ is equal for all j. The value of a flow is denoted by $v(f)$. Now we can state the following theorem.

Theorem 2 *Suppose we are given n points $P_1, \ldots, P_n \in \mathbb{R}^d$ with non-negative weights $w_i \geq 0$. Then, the origin $P^* = (0, \ldots, 0)$ is an optimal solution of the 1-median problem if and only if there exists a perfect flow f in $I(P_1, \ldots, P_n, w)$ such that $v(f) = \sum_{i=1}^{n} w_i$.*

Example 3 *Suppose we are given the following points: $P_1 = (-1, 3)$, $P_2 = (2, 2)$, $P_3 = (4, -1)$, $P_4 = (2, -2)$, $P_5 = (0, -2)$ and $P_6 = (-4, 1)$ with the weights $w_1 = 3$, $w_2 = 1$, $w_3 = 1$, $w_4 = 2$, $w_5 = 2$ and $w_6 = 3$. Then, the corresponding instance $I(P_1, \ldots, P_n, w)$ of the balancing flow problem admits a perfect flow f with $v(f) = \sum_{i=1}^{n} w_i$ (see Figure 1). Thus, we can conclude that the origin is a 1-median.*

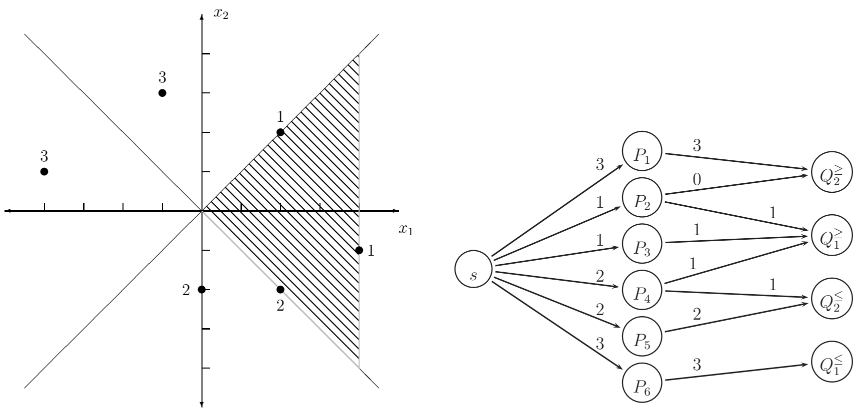

Fig. 1. On the left hand side we are given an instance of the 1-median problem where the cone $Q_1^{\geq}$ is highlighted. The right hand side shows the graph of the balancing flow problem (without the supersink t). On the edges a perfect flow is given.

It is easy to see that the inverse problem can be solved by the following algorithm:

Algorithm 1 *Step 1: Construct the instance* $I(P_1, \ldots, P_n, w)$
Step 2: Find a perfect flow f *in* $I(P_1, \ldots, P_n, w)$ *that maximizes* $v(f)$
Step 3: The new weights are given by $w_i^* = f(s, P_i)$

The main step is obviously the computation of a maximum perfect flow in Step 2. The maximum perfect flow problem can be reformulated as a parametric flow problem, where capacities on the edges $(Q_j^{\geq}, t)$ and $(Q_j^{\leq}, t)$ are given by a parameter λ_j. If we maximize

$$2 \sum_{j=1}^{d} \lambda_j$$

such that there exists a flow f that saturates all the edges entering the supersink t, the flow f is a maximum perfect flow. This maximization problem is closely related to parametric flow problems discussed in McCormick [3] and can indeed be solved in polynomial time.

References

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