Some bounds on alliances in trees

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1 Introduction

The study of alliances in graphs was first introduced by Hedetniemi, Hedetniemi and Kristiansen [4]. They introduced the concepts of defensive and offensive alliances, global offensive and global defensive alliances and studied alliance numbers of a class of graphs such as cycles, wheels, grids and complete graphs. Haynes et al. [2] studied the global defensive alliance numbers of different classes of graphs. They gave lower bounds for general graphs, bipartite graphs and trees, and upper bounds for general graphs and trees. Rodriquez-Velazquez and Sigarreta [8] studied the defensive alliance number and the global defensive alliance number of line graphs. A characterization of trees with equal domination and global strong defensive alliance numbers was given by Haynes, Hedetniemi and Henning [3]. Rodriguez-Velazquez and Sigarreta [5] gave bounds for the defensive, offensive, global defensive, global offensive alliance numbers in terms of the algebraic connectivity, the spectral radius, and the Laplacian spectral radius of a graph. They also gave bounds on the global offensive alliance number of cubic graphs in [6] and the global offensive alliance number for general graphs in [7].

Balakrishnan et al. [1] studied the complexity of global alliances. They showed that the decision problems for global defensive and global offensive alliances are both NP-complete for general graphs.

Given a simple graph $G = (V, E)$ and a vertex $v \in V$, the *open neighborhood* of v, $N(v)$, is defined as $N(v) = \{u : (u, v) \in E\}$. The *closed neighborhood* of v, denoted by $N[v]$, is $N[v] = N(v) \cup \{v\}$.

Definition 1 *A set $S \subset V$ is a* defensive alliance *if for every $v \in S$, $|N[v] \cap$*

$S| \geq |N(v) \cap (V - S)|$. *A defensive alliance S is called a* global defensive alliance *if S is also a dominating set.*

Definition 2 *A set $S \subset V$ is an* offensive alliance *if for every $v \in V - S$, $|N[v] \cap S| \geq |N[v] - S|$. An offensive alliance S is called a* global offensive alliance *if S is also a dominating set.*

Definition 3 *The* global defensive(offensive) alliance number *of G is the cardinality of a minimum size global defensive(offensive) alliance in G, and is denoted by $\gamma_a(G)(\gamma_o(G))$. A minimum size global defensive(offensive) alliance is called a $\gamma_a(G)$-set ($\gamma_o(G)$-set).*

In this paper, we study the global defensive and global offensive alliance numbers of trees. We find the asymptotic order of global defensive alliance number of complete k-ary trees, and compute exactly the global offensive alliance number. We also give a sharp bound on the difference between the global offensive and global defensive alliance numbers for a general tree.

The rest of the paper is organized as follows. In Section 2, we find the global defensive alliance number of complete binary and complete ternary trees. We also find tight bounds for the global defensive alliance number of complete k-ary trees, and determine the asymptotic order. In Section 3, we find the global offensive alliance number of complete k-ary trees. We also compare the global offensive and global defensive alliance numbers of a general tree.

2 Defensive Alliances in Complete k-ary Trees

A *k-ary tree* is a rooted tree where each node has at most k children. A *complete k-ary tree* is a k-ary tree in which all the leaves have the same depth and all the nodes except the leaves have k children. We let $T_{k,d}$ be the complete k-ary tree with depth/height d. The proofs of the following theorems are omitted.

Theorem 1 *Let n be the order of $T_{2,d}$. Then $\gamma_a(T_{2,d}) = \lceil \frac{2}{5}n \rceil$ for any d.*

Corollary 1 *If $d \equiv 2(mod\ 4)$ or $d \equiv 3(mod\ 4)$ then there is a unique $\gamma_a(T_{2,d})$-set. If $d \equiv 0(mod\ 4)$ or $d \equiv 1(mod\ 4)$ then there are exactly two $\gamma_a(T_{2,d})$-sets.*

Theorem 2 *If $d \geq 4$ then $\gamma_a(T_{3,d}) = \lfloor \frac{19}{36}n \rfloor$ if d is odd and $\gamma_a(T_{3,d}) = \lceil \frac{19}{36}n \rceil$ if d is even.*

Theorem 3 $\frac{27}{64}n - 2 \leq \gamma_a(T_{4,d}) \leq \frac{27}{64}n + 2$.

When k is large, the methods used to prove the above theorems are difficult to apply. Therefore, for general k, we give upper and lower bounds for $\gamma_a(T_{k,d})$.

Theorem 4 *For $d \geq 2$, and $k \geq 2$,*

$$k^{d-1}\left\lfloor \frac{k-1}{2} \right\rfloor + k^{d-1} + k^{d-2} \leq \gamma_a(T_{k,d}) \leq k^{d-1}\left\lfloor \frac{k-1}{2} \right\rfloor + k^{d-1} + k^{d-2} + k^{d-3}.$$

It follows that $\gamma_a(T_{k,d}) \sim k^{d-1}\left\lfloor \frac{k-1}{2} \right\rfloor$, where the asymptotics is taken to be in terms of k. Since the number of vertices of $T_{k,d}$ is $n = \frac{k^{d+1}-1}{k-1}$ we get $\gamma_a(T_{k,d}) \sim \frac{n}{2}$ when k tends to infinity. For offensive alliances we have the following result, the proof of which is omitted.

3 Offensive Alliances vs. Defensive Alliances in general trees

Theorem 5 *Let $T_{k,d}$ be the complete k-ary tree with depth $d \geq 1$. Then, $\gamma_o(T_{k,d}) = \left\lfloor \frac{n}{k+1} \right\rfloor$.*

Note that $\gamma_o(T_{k,d}) \sim \frac{n}{k}$ with respect to k. As k becomes very large the difference between $\gamma_a(T_{k,d})$ and $\gamma_o(T_{k,d})$ approaches $n/2$. In general, we are interested if this difference can be larger for other trees. In fact, we have the following theorem.

Theorem 6 *For any tree T of order n, $\gamma_a(T) \leq \gamma_o(T) + \frac{n}{2}$.*

Proof 1 *Root the tree T at a vertex of largest* eccentricity*(the eccentricity of a vertex x is equal to $max_{y \in V(G)} d(x, y)$). Let T have a depth d, and let v be a vertex at depth $d-2$. Let u be v's parent. We are going to proceed by induction on n. We may assume that $diam(T) \geq 3$. Otherwise, T is a star and the theorem holds (this also establishes the base case).*
Let T_v be the subtree of T rooted at vertex v. Let $T^{'} = T - T_v$ be the subtree of T obtained by removing all the vertices of T_v, and let $|T^{'}| = n^{'}$. Define P to be the set of children of v in T which are support vertices. Denote by L the set of children of v which are leaves. By assumption on the diameter of T, $|P| \geq 1$. Let y_i denote the number of children of each vertex in P, $1 \leq i \leq |P|$. The proofs of the following two claims are omitted due to space restrictions.

Claim 1 $\gamma_o(T^{'}) \leq \gamma_o(T) - |P|$.

Claim 2 $\gamma_a(T) \leq \gamma_a(T^{'}) + k$ *where* $k = 1 + |P| + max\left(\lceil \frac{|L|-|P|}{2} \rceil, 0\right) + \sum_{i=1}^{|P|} \lfloor \frac{y_i - 1}{2} \rfloor$.

By the last claim and the induction hypothesis we have

$$\gamma_a(T) \leq \gamma_a(T^{'}) + k \leq \gamma_o(T^{'}) + \frac{n^{'}}{2} + k.$$

What is left to prove is that $\gamma_o(T^{'}) + \frac{n^{'}}{2} + k \leq \gamma_o(T) + \frac{n}{2}$. By the first claim, it is sufficient to prove that $k - \lfloor \frac{n-n^{'}}{2} \rfloor \leq |P|$. Since $|P| \geq 1$, we have that

$$1+max\left(\left\lceil \frac{|L|-|P|}{2} \right\rceil, 0\right)+\sum_{i=1}^{|P|}\left\lfloor \frac{y_i-1}{2} \right\rfloor \leq 1+\left\lfloor \frac{|L|}{2} \right\rfloor+\sum_{i=1}^{|P|}\left\lfloor \frac{y_i-1}{2} \right\rfloor \leq \left\lfloor \frac{n-n^{'}}{2} \right\rfloor,$$

as required.

The above bound is best possible. Consider $K_{1,n-1}$ where n is odd. Then $\gamma_o(K_{1,n-1}) = 1$ and $\gamma_a(K_{1,n-1}) = 1 + \frac{n-1}{2}$.

In a bipartite graph, each partite set forms a global offensive alliance. It follows that $\gamma_o(T) \leq \frac{n}{2}$ for any tree T. Therefore, $|\gamma_a(T) - \gamma_o(T)| \leq \frac{n}{2}$. However, we believe the following stronger result is true: for any n-vertex tree T, $\gamma_o(T) \leq \gamma_a(T) + \frac{n}{6}$. This conjecture, if true, is essentially best possible because of the following theorem, the proof of which we omit.

Theorem 7 *For any constant $C > 0$, there exists an n-vertex tree T with $\gamma_o(T) \geq \gamma_a(T) + \frac{n}{6} - C$.*

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