# Cycle embedding in alternating group graphs with faulty vertices and faulty edges

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## 1 Introduction

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Unless otherwise stated, we follow [2] for graph terminologies and notations. A path $P_{v_0,v_k} = \langle v_0, v_1, \ldots, v_k\rangle$ is a sequence of distinct vertices except possibly $v_0 = v_k$ such that every two consecutive vertices are adjacent. The *length* of a path is the number of edges on the path. The *distance* between $u$ and $v$ is denoted by $d(u, v)$, which is the length of a shortest path between $u$ and $v$. A *cycle* is a special path with at least three vertices such that the first vertex is the same as the last one. A cycle of length $l$ is referred to as an *l-cycle*.

An interconnection network is usually modeled as an undirected simple graph, where the vertices represent processors and the edges represent communication links between processors. Study of the topological properties of an interconnection network is an important part of the study of any parallel or distributed system. The alternating group graph [8], which is an instance of Cayley graphs, is suitable to serve as a network because of its scalability and other favorable properties, e.g., regularity, recursiveness, symmetry, sublogarithmic degree and diameter, and maximal fault tolerance.

Let $u = a_1a_2 \cdots a_n$ be a permutation of 1, 2, ..., $n$, i.e., $a_i \in \{1, 2, \ldots, n\}$ and $a_i \neq a_j$ for $i \neq j$. A pair of symbols $a_i$ and $a_j$ of $u$ are said to be an inversion if $a_i < a_j$ whenever $i > j$. An even permutation is a permutation that contains an even number of inversions. Let $A_n$ denote the set of all even permutations

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over $\{1,2,\ldots,n\}$. For $3 \le i \le n$, we define two operations, $g_i^+$ and $g_i^-$, on $A_n$ by setting $ug_i^+$ (respectively, $ug_i^-$) to be the permutation obtained from $u$ by rotating the symbols $a_1$, $a_2$, $a_i$ from left to right (respectively, from right to left), while retaining the other $n-3$ symbols stationary. For example, we have $12345g_4^+ = 41325$ and $12345g_4^- = 24315$. The $n$-dimensional alternating group graph $AG_n$, has the vertex set $V(AG_n) = A_n$ and the edge set $E(AG_n) = \{(u,v)|u,v \in V(AG_n)$ and $v = ug_i^+$ or $v = ug_i^-$ for some $3 \le i \le n\}$. It is not difficult to see that $AG_n$ is regular with vertex degree $2(n-2)$, $|V(AG_n)| = n!/2$, and $|E(AG_n)| = (n-2)n!/2$. In addition, $AG_n$ is both vertex symmetric and edge symmetric [8].

For $n \ge 3$ and $1 \le i \le n$, let $A_n^{(i)}$ be the subset of $A_n$ that consists of all even permutations with element $i$ in the rightmost position, and let $AG_n^{(i)}$ be the subgraph of $AG_n$ induced by $A_n^{(i)}$. Obviously, $AG_n^{(i)}$ is isomorphic to $AG_{n-1}$ for every $i \in \{1,2,\ldots,n\}$. Due to the hierarchical structure, $AG_n$ can also be defined recursively as follows. $AG_n$ is constructed from $n$ disjoint copies of $(n-1)$-dimensional alternating group graphs $AG_n^{(i)}$ for $i \in \{1,2,\ldots,n\}$ such that $AG_n^{(i)}$ and $AG_n^{(j)}$, $i \ne j$, are connected by $(n-2)!$ edges, called *external edges*, of the form $(kj\cdots i, ik\cdots j)$ or $(jk\cdots i, ki\cdots j)$ for $k \in \{1,2,\ldots,n\} \setminus \{i,j\}$. By contrast, edges joining vertices in the same subgraph $AG_n^{(i)}$ are called *internal edges*. In particular, for each internal edge $(u,v)$ with $u = kj\cdots i$ and $v = jk'\cdots i$ in $AG_n^{(i)}$, there exist two adjacent vertices $s = ik\cdots j$ and $t = k'i\cdots j$ in $AG_n^{(j)}$ such that $s = ug_n^+$, $t = vg_n^-$, and $\langle u,s,t,v,u\rangle$ forms a 4-cycle in $AG_n$. For convenience, such a property is called the 4-*cycle structure* of $(u,v)$. Note that the pair of vertices $s$ and $t$ is uniquely determined by the 4-cycle structure of $(u,v)$. As a result, every vertex $u \in V(AG_n^{(i)})$ is connected to exactly 2 external edges and $2n-6$ internal edges.

A path (or cycle) in $G$ is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every vertex of $G$ exactly once. A graph $G$ is called *Hamiltonian* if it has a Hamiltonian cycle. $G$ is called *Hamiltonian-connected* if every two vertices of $G$ are connected by a Hamiltonian path. For an integer $r \ge 3$, $G$ is called *r-pancyclic* if it contains an $l$-cycle for each $l$ with $r \le l \le |V(G)|$. In particular, $G$ is called *vertex r-pancyclic* (or *edge r-pancyclic*) if every vertex (or edge) of $G$ belongs to an $l$-cycle for each $l$ with $r \le l \le |V(G)|$. A 3-pancyclic graph, a vertex 3-pancyclic graph, and an edge 3-pancyclic graph are called *pancyclic*, *vertex-pancyclic*, and *edge-pancyclic*, respectively. Exploring the pancyclicity of the given graph has attracted a lot of mathematicians [1,9,10]. Recently, some researchers have focused on the problem on interconnection networks because networks with cycle topology are suitable for designing simple algorithms with low communication costs (for example, [5,6,13,16]).

A graph $G$ is *panconnected* if, for any two distinct vertices $u,v \in V(G)$ and for each integer $l$ with $d(u,v) \le l \le |V(G)|-1$, there is a $P_{u,v}$ of length $l$ in $G$. It was shown in [4] that the alternating group graph is panconnected.

**Lemma 1** ([4]) *For $n \geq 3$, $AG_n$ is panconnected.*

Since faults may occur in networks, the consideration of fault-tolerance ability is a major factor in evaluating the performance of networks. Cycle embedding is also concerned extensively in many interconnection networks with faulty elements [7,11,14]. Suppose $F_v \subset V(G)$, $F_e \subset E(G)$, and $F = F_v \cup F_e$. A graph $G$ is *k-vertex-fault-tolerant pancyclic* if $G - F_v$ remains pancyclic whenever $|F_v| \leq k$. A graph $G$ is called *k-edge-fault-tolerant pancyclic* if $G - F_e$ is pancyclic whenever $|F_e| \leq k$. A graph $G$ is called *k-fault-tolerant pancyclic* if $G - F$ remains pancyclic when $|F| \leq k$. The notion for *k-fault-tolerant Hamiltonian*, *k-fault-tolerant Hamiltonian-connected*, *k-fault-tolerant vertex r-pancyclic*, and *k-fault-tolerant edge r-pancyclic* can also be defined similarly.

**Lemma 2** ([3]) *$AG_n$ is $(n-4)$-vertex-fault-tolerant edge 4-pancyclic and $(n-3)$-vertex-fault-tolerant vertex-pancyclic, where $n \geq 4$.*

**Lemma 3** ([15]) *$AG_n$ is $(2n-6)$-vertex-fault-tolerant pancyclic, where $n \geq 3$.*

**Lemma 4** ([12]) *$AG_n$ is $(2n-6)$-edge-fault-tolerant pancyclic, where $n \geq 3$.*

**Lemma 5** ([12]) *$AG_n$ is $(2n-7)$-fault-tolerant Hamiltonian-connected, where $n \geq 4$.*

## 2 Main results

We improve previous results in Lemma 2, Lemma 3, and Lemma 4, by showing that the $n$-dimensional alternating group graph $AG_n$ is $(n-4)$-fault-tolerant edge 4-pancyclic, $(n-3)$-fault-tolerant vertex-pancyclic, and $(2n-6)$-fault-tolerant pancyclic, while considering both faulty vertices and faulty edges. All the results we achieved here are optimal with respect to the number of faulty elements tolerated.

**Theorem 1** *$AG_n$ is $(n-4)$-fault-tolerant edge 4-pancyclic, where $n \geq 4$.*

**Theorem 2** *$AG_n$ is $(n-3)$-fault-tolerant vertex-pancyclic, where $n \geq 3$.*

**Theorem 3** *$AG_n$ is $(2n-6)$-fault-tolerant pancyclic, where $n \geq 3$.*

Since a graph is Hamiltonian if it is pancyclic, we have the following Corollary.

**Corollary 1** *$AG_n$ is $(2n-6)$-fault-tolerant Hamiltonian, where $n \geq 3$.*

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