On the partition dimension of Cartesian product graphs

Ismael G. Yero [a], Juan A. Rodríguez-Velázquez [a] and Magdalena Lemańska [b]

[a] *Department of Computer Engineering and Mathematics Rovira i Virgili University, Av. Països Catalans 26, 43007 Tarragona, Spain*

[b] *Department of Technical Physics and Applied Mathematics Gdansk University of Technology, ul. Narutowicza 11/12 80-233 Gdansk, Poland*

Abstract

Let $G = (V, E)$ be a connected graph. The distance between two vertices $u, v \in V$, denoted by $d(u, v)$, is the length of a shortest $u-v$ path in G. The distance between a vertex $v \in V$ and a subset $P \subset V$ is defined as $min\{d(v, x) : x \in P\}$, and it is denoted by $d(v, P)$. An ordered partition $\{P_1, P_2, ..., P_t\}$ of vertices of a graph G, is a *resolving partition* of G, if all the distance vectors $(d(v, P_1), d(v, P_2), ..., d(v, P_t))$ are different. The *partition dimension* of G, denoted by $pd(G)$, is the minimum number of sets in any resolving partition of G. In this article we show that for all pair of connected graphs G, H, $pd(G \times H) \leq pd(G) + pd(H)$ and $pd(G \times H) \leq pd(G) + dim(H)$. Consequently, we show that $pd(G \times H) \leq dim(G) + dim(H) + 1$.

Key words: Resolving sets, resolving partition, partition dimension, Cartesian product.

1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [9] and Slater [16], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [2–8,14]. Also, Slater described the usefulness of these ideas into long range aids to navigation [16]. Recently, these concepts were used by a pharmacy company while attempting to develop a capability of large datasets of chemical graphs [12,13]. Other applications of this concept to navigation of robots in networks and other areas appear in [5,11,14]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [15], locating domination [10], resolving domination [1] and resolving partitions [4,7,8].

Email address: `ismael.gonzalez@urv.cat` (Ismael G. Yero).

Given a graph $G = (V, E)$ and a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of G, the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$, where $d(v, v_i)$, with $1 \leq i \leq k$, denotes the distance between the vertices v and v_i. We say that S is a *resolving set* of G if for every pair of vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension*[1] of G is the minimum cardinality of any resolving set of G, and it is denoted by $dim(G)$. The metric dimension of graphs is studied in [2–6,17]. Given an ordered partition $\Pi = \{P_1, P_2, ..., P_t\}$ of the vertices of G, the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), ..., d(v, P_t))$, where $d(v, P_i)$, with $1 \leq i \leq t$, represents the distance between the vertex v and the set P_i, that is $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$. We say that Π is a *resolving partition* of G if for every pair of vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G and it is denoted by $pd(G)$. The partition dimension of graphs is studied in [4,7,8,17]. It is natural to think that the partition dimension and metric dimension are related; in [7] it was shown that for any nontrivial connected graph G we have

$$pd(G) \leq dim(G) + 1. \tag{1}$$

The study of relationships between invariants of Cartesian product graphs and invariants of its factors appears frequently in research about graph theory. In the case of resolvability, the relationships between the metric dimension of the Cartesian product graphs and the metric dimension of its factors was studied in [2,3]. An open problem on the dimension of Cartesian product graphs is to prove (or finding a counterexample) that for all pair of graphs G, H; $dim(G \times H) \leq dim(G) + dim(H)$. In the present paper we study the case of resolving partition in Cartesian product graphs, by giving some relationships between the partition dimension of Cartesian product graphs and the partition dimension of its factors. More precisely, we show that for all pair of connected graphs G, H; $pd(G \times H) \leq pd(G) + pd(H)$ and $pd(G \times H) \leq pd(G) + dim(H)$. Consequently, we show that $pd(G \times H) \leq dim(G) + dim(H) + 1$.

2 Results

Theorem 1 *For any connected graphs G_1 and G_2,*

$$pd(G_1 \times G_2) \leq pd(G_1) + pd(G_2).$$

By (1) we obtain the following direct consequence of Theorem 1.

Corollary 2 *For any connected graphs G_1 and G_2,*

$$pd(G_1 \times G_2) \leq pd(G_1) + dim(G_2) + 1.$$

As we can see below, the above relationship can be improved.

[1] Also called locating number.

Theorem 3 *For any connected graphs G_1 and G_2,*

$$pd(G_1 \times G_2) \leq pd(G_1) + dim(G_2).$$

We note that there are graphs for which Theorem 1 estimates $pd(G_1 \times G_2)$ better than Theorem 3 and vice versa. For example Theorem 1 leads to $pd(K_n \times P_n) \leq n + 2$ while Theorem 3 gives $pd(K_n \times P_n) \leq n + 1$. On the contrary, if G denotes the graph in Figure 1, Theorem 1 leads to $pd(G \times G) \leq 8$ while Theorem 3 gives $pd(G \times G) \leq 13$.

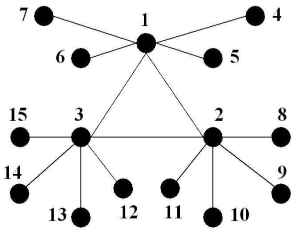

Fig. 1. $\{\{1, 4, 8, 12\}, \{2, 5, 9, 13\}, \{3, 6, 10, 14\}, \{7, 11, 15\}\}$ is a resolving partition of G and $\{4, 5, 6, 8, 9, 10, 12, 13, 14\}$ is a resolving set of G.

As a direct consequence of above theorem and (1) we deduce the following interesting result.

Corollary 4 *For any connected graphs G_1 and G_2,*

$$pd(G_1 \times G_2) \leq dim(G_1) + dim(G_2) + 1.$$

One example of graphs for which the equality holds in Corollary 4 (and also in Corollary 5 (ii)) are the graphs belonging to the family of grid graphs: $pd(P_r \times P_t) = 3$.

Corollary 5 *For any connected graph G,*

(i) $pd(G \times K_n) \leq pd(G) + n - 1$.
(ii) $pd(G \times P_n) \leq pd(G) + 1$.
(iii) $pd(G \times C_n) \leq pd(G) + 2$.
(iv) $pd(G \times K_{1,n}) \leq pd(G) + n - 1$.

Acknowledgments: This work was partly supported by the Spanish Ministry of Science and Innovation through projects TSI2007-65406-C03-01 "E-AEGIS", CONSOLIDER INGENIO 2010 CSD2007-0004 "ARES".

References

[1] R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, *Mathematica Bohemica* **128** (1) (2003) 25–36.

[2] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of Cartesian product of graphs, *SIAM Journal of Discrete Mathematics* **21** (2) (2007) 273–302.

[3] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Mathematics* **22** (2005) 129–133.

[4] G. Chappell, J. Gimbel, C. Hartman, Bounds on the metric and partition dimensions of a graph, *Ars Combinatoria* **88** (2008) 349–366.

[5] G. Chartrand, L. Eroh, M. A. Jhonson, O. R. Oellermann, Resolvabiblity in graphs ans the metric dimension of a graph, *Discrete Applied Mathematics* **105** (2000) 99–113.

[6] G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, *Computer and Mathematics with Appplications* **39** (2000) 19–28.

[7] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, *Aequationes Mathematicae* **59** (2000) 45–54.

[8] M. Fehr, S. Gosselin, O. R. Oellermann, The partition dimension of Cayley digraphs *Aequationes Mathematicae* **71** (2006) 1–18.

[9] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria* **2** (1976) 191–195.

[10] T. W. Haynes, M. Henning, J. Howard, Locating and total dominating sets in trees, *Discrete Applied Mathematics* **154** (2006) 1293–1300.

[11] B. L. Hulme, A. W. Shiver, P. J. Slater, A Boolean algebraic analysis of fire protection, *Algebraic and Combinatorial Methods in Operations Research* **95** (1984) 215–227.

[12] M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *Journal of Biopharmaceutical Statististics* **3** (1993) 203–236.

[13] M. A. Johnson, Browsable structure-activity datasets, *Advances in Molecular Similarity* (R. Carb–Dorca and P. Mezey, eds.) JAI Press Connecticut (1998) 153–170.

[14] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Applied Mathematics* **70** (1996) 217–229.

[15] V. Saenpholphat, P. Zhang, Conditional resolvability in graphs: a survey, *International Journal of Mathematics and Mathematical Sciences* **38** (2004) 1997–2017.

[16] P. J. Slater, Leaves of trees, Proc. 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, *Congressus Numerantium* **14** (1975) 549–559.

[17] I. Tomescu, Discrepancies between metric and partition dimension of a connected graph, *Discrete Mathematics* **308** (2008) 5026–5031.